Lec 2: 01/21/2010

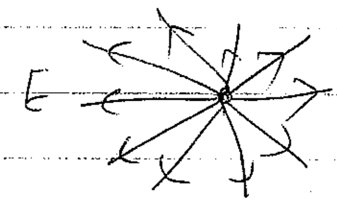
Friedmann-Robertson-Walker Universes

Isotropy and Homogeneity of the universe:

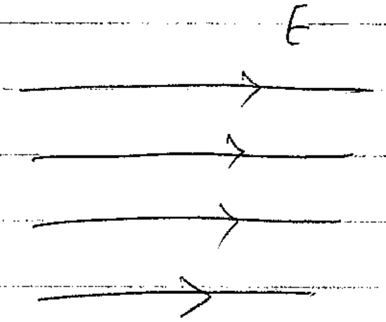
The observable universe is lumpy at small scales. However, at large scales it seems highly isotropic and homogeneous. The best evidence for isotropy comes from CMB. The temperature anisotropy in the CMB is of order of 1 in 10^5.

Of course, isotropy about one point (Earth, where we live and observe) does not imply homogeneity. Evidence from homogeneity comes from galaxy surveys like SDSS and 2dF that look at structures within a distance range of few billion light years. At distances larger than 100 Mpc (1 pc = 3.26 light years) the universe is homogeneous to a good degree.

Note that homogeneity alone does not mean isotropy (see the following examples):



isotropy about one point, no homogeneity



homogeneous, not isotropic

But isotropy about one point (CMB) and homogeneity (galaxy surveys) imply that the universe is both homogeneous and isotropic. I.E., that all points in the universe are exactly the same.

The question is how we can describe the evolution of a homogeneous and isotropic universe. The relevant theory is Einstein's general relativity. According to general relativity, matter and energy distribution determines the geometry of spacetime. This is very different from Newtonian dynamics and special relativity where spacetime is a static background in which the "events" are defined, and a sequence of events is related to dynamical evolution of a system.

Equivalence Principle:

In general relativity, spacetime is not static. Its geometry depends on the distribution of matter and energy. According to John Wheeler, the matter tells spacetime how to bend, and the spacetime tells matter how to move.

In this picture, gravity has a geometrical interpretation. A massive object bends the space around itself and the trajectory of a test particle passing by the object is deflected as a result of this bending, hence gravitational attraction.

This can be motivated heuristically by using the "Equivalence Principle". According to Einstein, this is the complete equivalence between a gravitational field and an accelerating reference frame.

This can be seen by comparing what an observer in a uniform gravitational field (Newton's gravity) and an observe in a frame with constant acceleration observe.

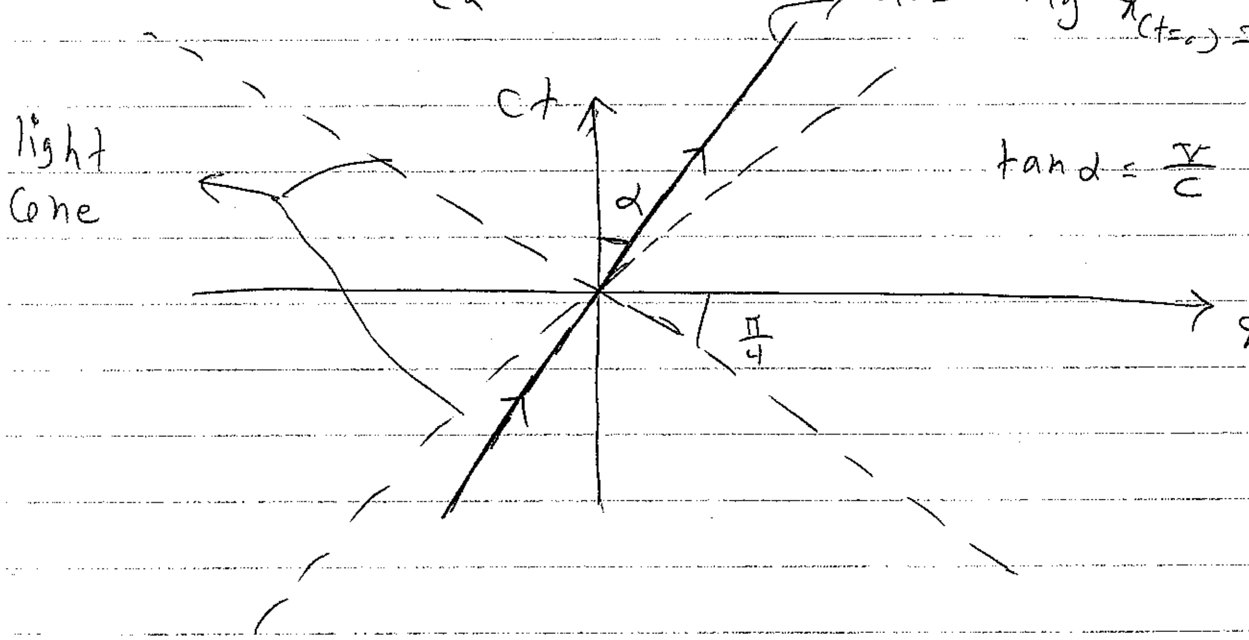
A particle under no force (but gravitation) moves with a constant acceleration g in the gravitational field. This also happence for a free particle in a frame with accelerating -g.

Now lets Consider the trajectory of a free particle in an inertial frame (no acceleration) and a frame with constant acceleration.

We consider one spatial dimension. In the spacetime, the trajectory of particle in an inertial frame is found from the equation of motion:

$$p = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}, \quad F = \frac{dp}{dt} = 0 \Rightarrow \boxed{v = \frac{dx}{dt} = \text{Const.}}$$

assuming $x_{(t=0)} = 0$

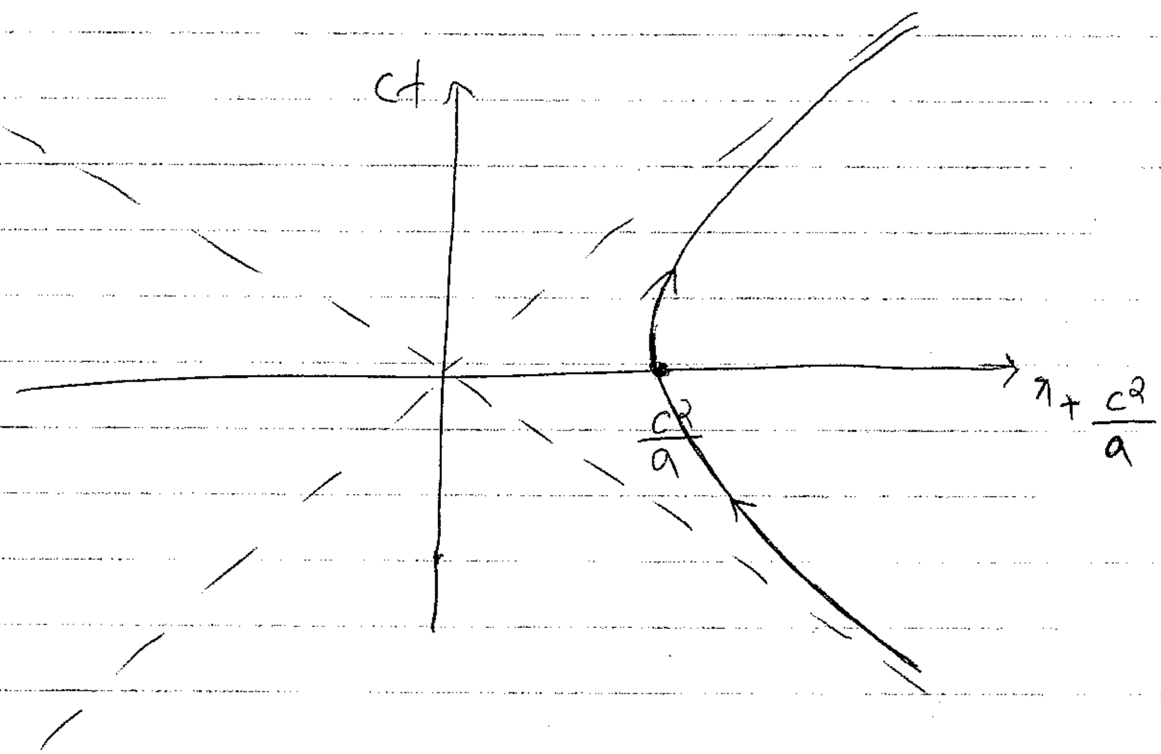


In an accelerating frame, a free particle has a constant acceleration "a". The equation of motion is:

$$\frac{dP}{dt} = ma \Rightarrow \frac{d}{dt}\left(\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}\right) = a \Rightarrow \boxed{x = \sqrt{c^2t^2 + \frac{c^4}{a^2}} - \frac{c^2}{a}}$$

assuming $x(t=0) = 0$
$v(t=0) = 0$

This is a hyperbola in the spacetime:



The trajectory of a free particle that was a straight line before has now transformed into a hyperbola. This may be interpretted as the bending of the spacetime as a result of acceleration.

Using the equivalence principle, one may conclude that in a gravitational field (due to the presence of mass/energy) spacetime bends as well.

How the geometry of spacetime is related to the distribution of matter/energy is governed by Einstein field equations:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \left(+ g_{\mu\nu}\Lambda\right)$$

Λ: Cosmological Constant

$T_{\mu\nu}$: Energy-Momentum tenser

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

$G_{\mu\nu}$: Einstein tensor

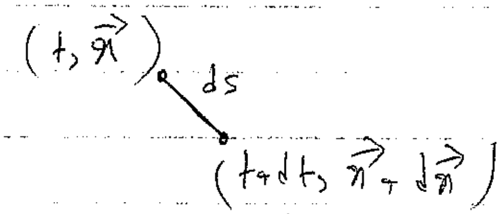
$R_{\mu\nu}$: Ricci ~~Curva~~ Curvature tensor

We will not discuss this equation except for the solutions for a homogeneous and isotropic distribution of matter/energy.

<u>Homogeneous and Istropic Spaces in 3 Dimensions:</u>

Since we live in three spatial dimensions (at least those that are large) we consider 4-dimensional spacetime. The distance between two points (t, x^1, x^2, x^3) and $(t+dt, x^1+dx^1, x^2+dx^2, x^3+dx^3)$ is given by:

$$ds^2 = g_{ij}\, dx^i dx^j + g_{00}\, dt^2$$
$(1 \leq i, j \leq 3)$



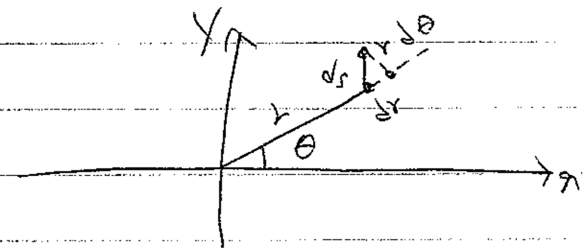
Note that we have used $c=1$ (natural units) here.

Here we focus on the spatial part of the metric g_{ij} ($\overset{0 \leq \nu, \mu \leq 3}{g_{\nu\mu}}$ is a 4×4 matrix, g_{ij} is 3×3).

It turns out from Einstein's equations that for a homogeneous distribution of matter/energy the spacelike hypersurface (x^1, x^2, x^3) is also homogeneous and isotropic.

There are three classes of 3-dimensional homogeous and isotropic spaces. For simplicity, lets start with the 2-dimensional case.

(1) Two-dimensional plane $\mathbb{R}^2$ is clearly homogeous and isotropic. In Cartesian coordinates, the distance between two nearby points (x, y) and $(x+dx, y+dy)$ is given by:



$$ds^2 = dx^2 + dy^2$$

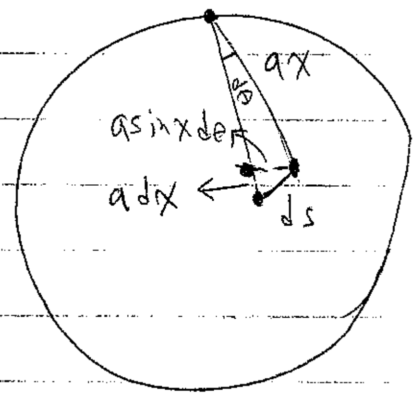
In polar coordinates (r, θ) we have:

$$ds^2 = dr^2 + r^2 d\theta^2 = a^2\left(d\chi^2 + \chi^2 d\theta^2\right)$$

Here we have scaled r by an overall factor a such that $r = a\chi$.

(2) Two-dimensional sphere S^2 is also homogeous and isotropic. Choosing the north pole as the origin, the distance between two nearby points is given as;

$ds^2 = a^2 d\chi^2 + a^2 \sin^2\chi \, d\theta^2$



Here a is the radius of the sphere and χ is the azimuth angle. We therefore have:

$$ds^2 = a^2 (d\chi^2 + \sin^2\chi \, d\theta^2)$$

(3) Two dimensional H^2 hyperbola is also homogeneous and isotropic. The distance between two nearby points on a hyperbola is given by:

$$ds^2 = a^2 (d\chi^2 + \sinh^2\chi \, d\theta^2)$$

Going to three dimensions, we have:

$$ds^2 = a^2 [d\chi^2 + \chi^2 (d\theta^2 + \sin^2\theta \, d\phi^2)] \qquad \mathbb{R}^3$$

$$ds^2 = a^2 [d\chi^2 + \sin^2\chi (d\theta^2 + \sin^2\theta \, d\phi^2)] \qquad \mathbb{S}^3$$

$$ds^2 = a^2 [d\chi^2 + \sinh^2\chi (d\theta^2 + \sin^2\theta \, d\phi^2)] \qquad \mathbb{H}^3$$

In all the three cases a is an overall scale factor.

(17)

Its usefulness becomes clear when we start discussing the evolution of a homogeneous and isotropic universe according to general relativity.

Another way to write down the metric for the 3-dimensional homogeneous and isotropic spaces is by making the following change of variables:

$r = \chi \quad R^3, \quad r = \sin\chi \quad S^3, \quad r = \sinh\chi \quad H^3$

Then we find:

$$ds^2 = a^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

$$k = \begin{cases} +1 & S^3 \text{ (closed)} \\ 0 & R^3 \text{ (flat)} \\ -1 & H^3 \text{ (open)} \end{cases}$$